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Periodic Double Obstacle Problems and Applications

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§1. Introduction

Recently, we have shown that there exists a time-periodic global attractor for time-periodic dynamical systems governed by subdifferentials in Hilbert spaces (cf. [3]). But we do not know the large-time behaviour of each solution. In general, the solution does not converge to any periodic solution, although the system is time-periodic (cf. [6, 7]).

In this paper we consider time-periodic double obstacle problems in order to show that solutions are asymptotically periodic, if given obstacle functions are periodic in time.

At first, we consider a scalar T_0-periodic double obstacle problem of the form:

$$u'(t)+\partial I_{K(t)}(u(t))+g(u(t))\ni 0, \quad t\geq 0, \tag{1.1}$$

where for each $t\geq 0$ and given T_0-periodic obstacle functions $\sigma_0,\ \sigma_1$ on $R_+:=[0,+\infty)$

$$K(t):=\{z\in R;\ \sigma_0(t)\leq z\leq\sigma_1(t)\},$$

$\partial I_{K(t)}$ is a subdifferential of the indicator function $I_{K(t)}(\cdot)$ on R and g is a smooth function on R which is in general non-monotone on R such as $g(u)=u^3-u$.

In this case, we shall show that any solution of (1.1) is asymptotically T_0-periodic. Namely, for any solution u of (1.1) there is a T_0-periodic solution u_p of (1.1) such that

$$u(t)-u_p(t)\longrightarrow 0 \quad \text{as } t\rightarrow+\infty.$$

Next, we give two applications of our result on scalar T_0-periodic obstacle problems. In the first application we discuss the asymptotically T_0-periodicity of the solution of a Stefan problem with hysteresis in the higher dimensional case which is left unsolved in [8].

In the second application we consider a partial differential equation with T_0-periodic double obstacles of the form:

$$u'-\kappa\Delta u+g(u)+\partial I_{K(t)}(u(t))\ni 0 \quad \text{in } Q:=R_+\times\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Sigma := R_+ \times \Gamma, \tag{1.3}$$

where Ω is a bounded domain in R^N $(1 \leq N < +\infty)$, with smooth boundary $\Gamma := \partial\Omega$, for each $t \in R_+ := [0, +\infty)$ and given obstacle functions σ_0, σ_1, $K(t)$ is the set

$$\left\{ z \in L^2(\Omega);\ \sigma_0(t,\cdot) \leq z \leq \sigma_1(t,\cdot) \quad \text{a.e. on } \Omega \right\},$$

$\partial I_{K(t)}$ is the subdifferential of the indicator function $I_{K(t)}$ on $L^2(\Omega)$ and g is a non-monotone smooth function on R. Under some assumptions, we shall show that solutions of (1.2)-(1.3) are asymptotically T_0-periodic.

§2. Scalar double obstacle problems

Let $0 < T_0 < +\infty$ be fixed and we assume that given obstacle functions σ_0, $\sigma_1 \in W^{1,2}(R_+)$ satisfy the following conditions:

$$\sigma_0 \leq \sigma_1 \quad \text{on } R_+, \tag{2.1}$$

$$\sigma_0(t) = \sigma_0(t + T_0) \quad \text{and} \quad \sigma_1(t) = \sigma_1(t + T_0) \quad \text{for any } t \geq 0. \tag{2.2}$$

For each time $t \geq 0$, we define the closed set $K(t)$ and proper l.s.c. convex function $I_{K(t)}$ on R, respectively, by

$$K(t) := \{z \in R; \sigma_0(t) \leq z \leq \sigma_1(t)\} \tag{2.3}$$

and

$$I_{K(t)}(z) := \begin{cases} 0 & \text{if } z \in K(t), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Now let us consider an ordinary differential equation with T_0-periodic double obstacle of the form

$$u'(t) + \partial I_{K(t)}(u(t)) + g(u(t)) \ni 0, \quad t \geq 0, \tag{2.5}$$

where $\partial I_{K(t)}$ is the subdifferential of $I_{K(t)}(\cdot)$ and g is a non-monotone smooth function on R, in general.

Definition 2.1. (1) A function $u : R_+ \to R$ is called a solution of (2.5), if it satisfies the following conditions (C1)-(C3):

(C1) $u \in W^{1,2}_{loc}(R_+)$.

(C2) $u(t) \in K(t)$ for any $t \in R_+$.

(C3) There exists a function $\xi \in L^2_{loc}(R_+)$ such that

$$\xi(t) \in \partial I_{K(t)}(u(t)) \quad \text{for a.e. } t \in R_+$$

and

$$u'(t) + \xi(t) + g(u(t)) = 0 \qquad \text{for a.e.} \quad t \in R_+.$$

(2) A function $u : R_+ \to R$ is called a solution of the Cauchy problem for (2.5), if u is a solution of (2.5) and satisfies the initial condition:

$$u(0) = u_0.$$

(3) A function $u : R_+ \to R$ is called a T_0-periodic solution of (2.5), if u is a solution of (2.5) and satisfies the T_0-periodic condition:

$$u(t + T_0) = u(t) \quad \text{for any } t \geq 0.$$

We can easily see that (2.5) is reformulated as an evolution equation governed by time-dependent subdifferentials of the form

$$\text{(E)} \qquad u'(t) + \partial\varphi^t(u(t)) + g(u(t)) \ni 0 \text{ in } H, \quad t > 0,$$

where H is a real Hilbert space, $\partial\varphi^t$ is the subdifferentials of time-dependent convex function $\varphi^t(\cdot)$ on H and $g(\cdot)$ is a Lipschitz operator on H. In fact, we take R as the Hilbert space H and $I_{K(t)}(\cdot)$ as $\varphi^t(\cdot)$. By (2.2), we easily see that the class $\{\varphi^t\} := \{\varphi^t; t \in R_+\}$ of proper l.s.c. convex functions φ^t on H satisfies T_0-periodicity condition

$$\varphi^{t+T_0}(\cdot) = \varphi^t(\cdot) \quad \text{on } H, \quad \forall t \in R_+.$$

Hence, by applying the abstract results in [3] we get the existence-uniqueness and global boundedness results of the solution of the Cauchy problem for (2.5).

As a main result on the asymptotic behaviour of solution u of (2.5), we have the following theorem.

Theorem 2.1. *Assume that $g(\xi) = 0$ has a finite number of roots. Then any solution u of (2.5) is asymptotically T_0-periodic, more precisely, one of the following four cases (1), (2), (3) and (4) occurs:*
(1) $u(t) - u^*(t) \to 0$ as $t \to +\infty$, where u^* is the maximal T_0-periodic solution of (2.5).
(2) $u(t) - u_*(t) \to 0$ as $t \to +\infty$, where u_* is the minimal T_0-periodic solution of (2.5).
(3) There is a root ξ_0 of $g(\xi) = 0$ such that $u(t) \to \xi_0$ as $t \to +\infty$.
(4) $u(t) - u_p(t) \to 0$ as $t \to +\infty$, where u_p is the unique T_0-periodic solution of (2.5).

By using some numerical experiences, we shall explain Theorem 2.1.
For simplicity, we assume that $g(u) = u^3 - u$, namely, there are three roots of $g(\xi) = 0$.
Now, we consider the following six obstacle cases.

<u>Case 1.</u> We assume that

$$\sigma_0(t) \leq -1 \quad \text{and} \quad 1 \leq \sigma_1(t), \quad \forall t \in R_+.$$

In this case, any solution u of (2.5) converges to one of stationary solutions -1, 0, 1 of (2.5) as $t \to +\infty$. The behaviour of solution u of (2.5) is illustrated in the Fig.2.1.

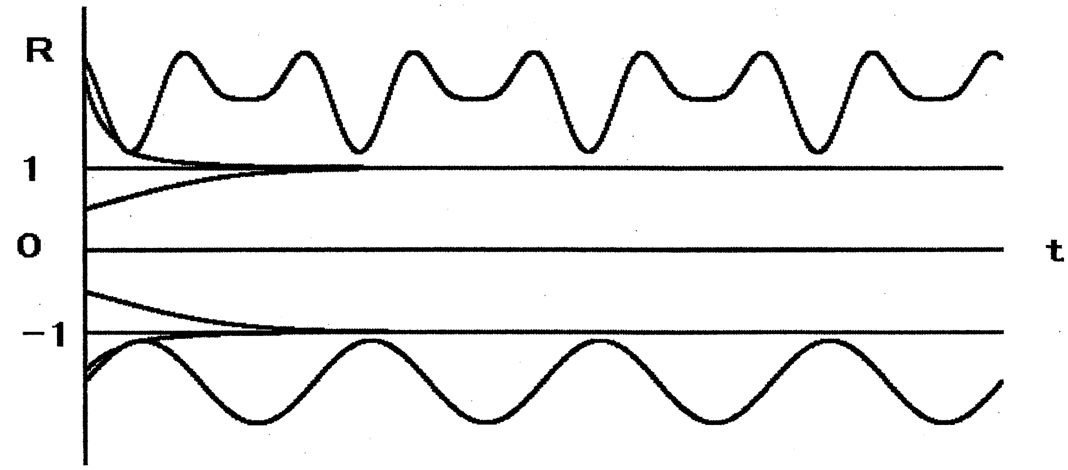

Fig.2.1

<u>Case 2.</u> Assume that $\sigma_1(t) \geq 0$ for any $t \in R_+$,

$$\sigma_0(t) \leq -1, \quad \forall t \in R_+ \quad \text{and} \quad \sigma_1(t_0) < 1 \quad \text{for some } t_0 \in R_+.$$

In this case, any solution u with initial data $u_0 > 0$ converges to the maximal T_0-periodic solution of (2.5). In fact, the solution u coincide with the maximal T_0-periodic solution of (2.5) after a certain finite time $t_1 \in R_+$. For the other data, the solution u converges to 0 or -1 as $t \to +\infty$. The behaviour of solution u of (2.5) is illustrated in the Fig.2.2.

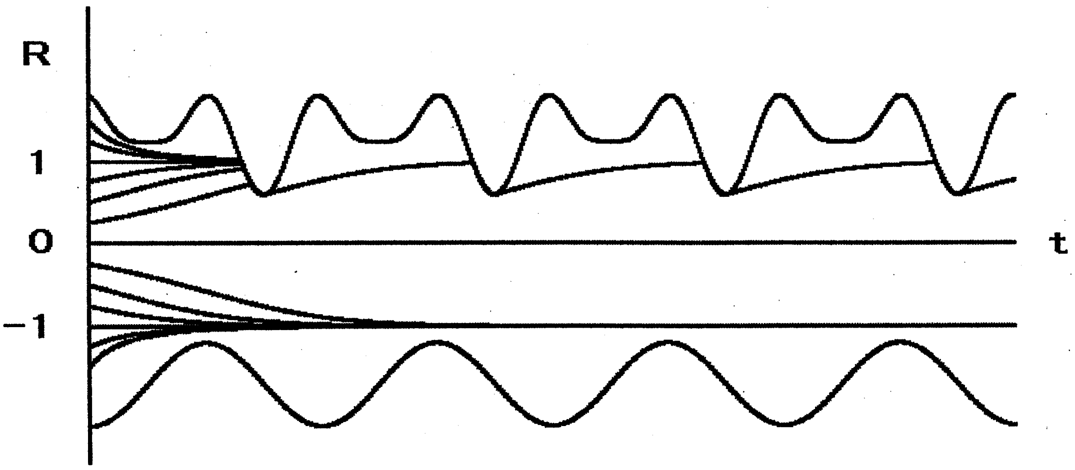

Fig.2.2

<u>Case 3.</u> Assume that $\sigma_0(t) < 0 \leq \sigma_1(t)$ for any $t \in R_+$,

$$-1 < \sigma_0(t_0) \quad \text{for some } t_0 \in R_+ \quad \text{and} \quad \sigma_1(t_1) < 1 \quad \text{for some } t_1 \in R_+.$$

In this case, for any solution u of (2.5) with initial data $u_0 > 0$ (resp. $u_0 < 0$) there is a finite time $t_2 \in R_+$ such that

$$u(t_2) = \sigma_1(t_2) \quad (\text{resp. } \sigma_0(t_2)).$$

Therefore, the solution u coincides with a maximal T_0-periodic solution u^* (resp. a minimal T_0-periodic solution u_*) of (2.5) after a certain finite time. If initial data $u_0 = 0$, the solution $u(t)$ is constant 0.

The behaviour of solution u of (2.5) is illustrated in the Fig.2.3.

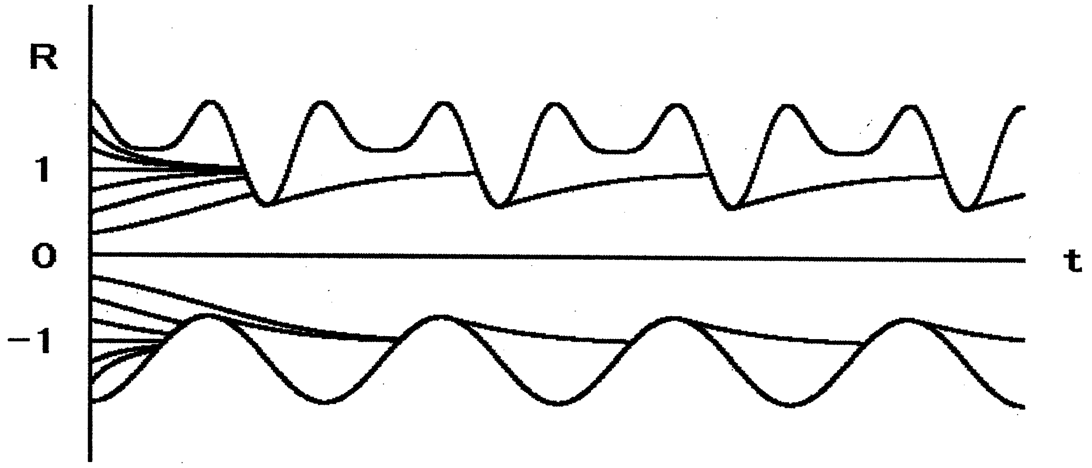

Fig.2.3

<u>Case 4.</u> Assume that

$$\sigma_0(t_0) \leq -1, \quad \forall t \in R_+ \quad \text{and} \quad \sigma_1(t_0) < 0 \quad \text{for some } t_0 \in R_+.$$

In this case, any solution u of (2.5) converges to a stationary solution -1 of (2.5) as $t \to +\infty$. The behaviour of solution u of (2.5) is illustrated in the Fig.2.4.

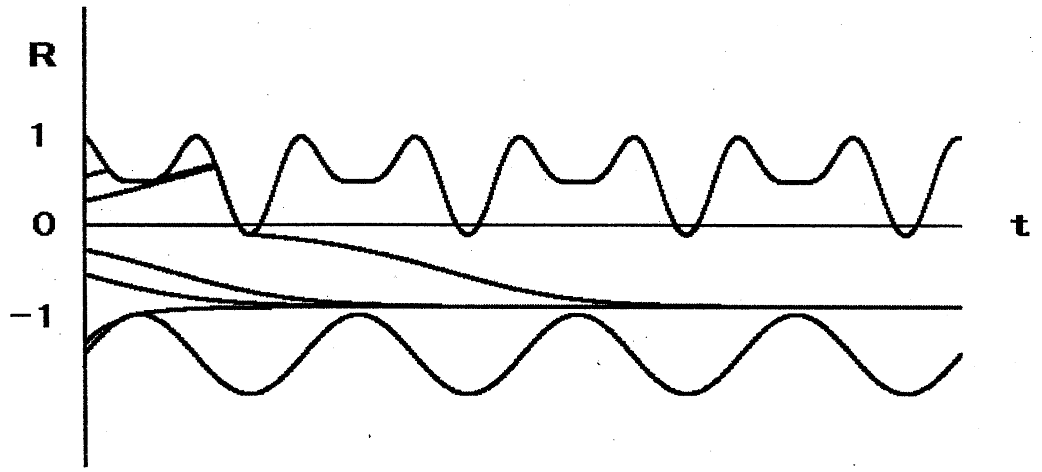

Fig.2.4

<u>Case 5.</u> Assume that $\sigma_0(t) < 0$ for any $t \in R_+$,

$$-1 < \sigma_0(t_0) \quad \text{for some } t_0 \in R_+ \quad \text{and} \quad \sigma_1(t_1) < 0 \quad \text{for some } t_1 \in R_+.$$

In this case, any solution u of (2.5) is negative somewhere. Therefore u converges to the unique T_0-periodic solution u_p of (2.5) as $t \to +\infty$. In fact, any solution u of (2.5) coincides with u_p after some finite time.

The behaviour of solution u of (2.5) is illustrated in the Fig.2.5.

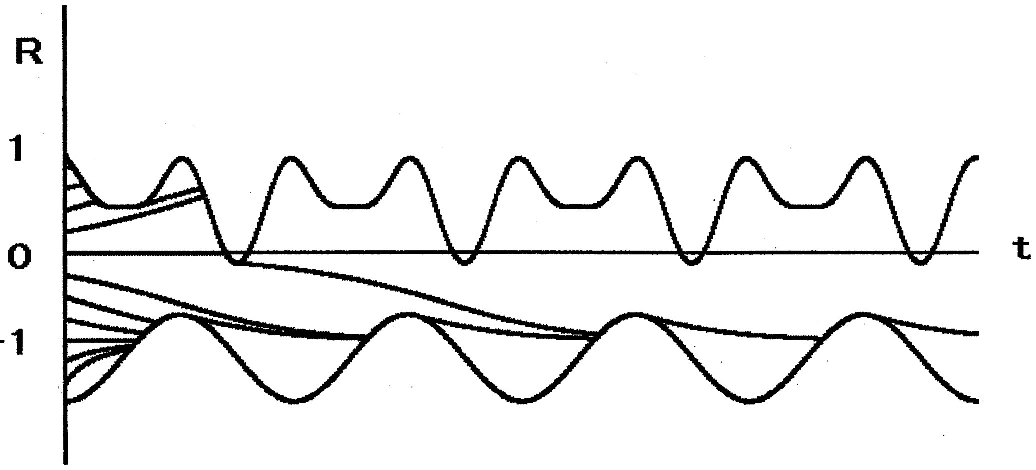

Fig.2.5

<u>Case 6.</u> Assume that

$$0 \leq \sigma_0(t_0) \quad \text{for some } t_0 \in R_+ \quad \text{and} \quad \sigma_1(t_1) \leq 0 \quad \text{for some } t_1 \in R_+.$$

In this case, it follows from the facts of Case 2-4 that there exists a unique T_0-periodic solution u_p of (2.5) and any solution u of (2.5) coincide with the unique T_0-periodic solution u_p of (2.5) after some finite time.

The behaviour of solution u of (2.5) is illustrated in the Fig.2.6.

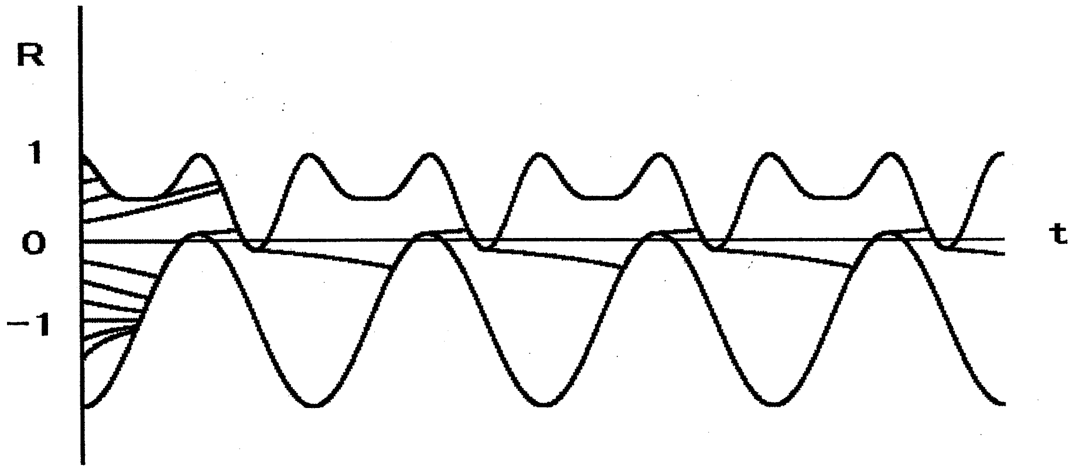

Fig.2.6

Remark. All the cases of relationships between σ_0 and σ_1 are covered by Cases 1-6, except their symmetric case.

§3. Application to a Stefan Problem with hysteresis

In this section, we consider a Stefan problem with hysteresis, which is a model for solid-liquid phase transition with superheating and undercooling effect.

In [8], the following system was treated:

$$[\theta + w]_t - \Delta\theta = f(t,x) \quad Q := (0,+\infty) \times \Omega, \tag{3.1}$$

$$w_t(t,x) + \partial I_{\theta(t,x)}(w(t,x)) \ni 0, \quad (t,x) \in Q, \tag{3.2}$$

$$\theta = g(x) \quad \text{on } \Sigma := (0,+\infty) \times \Gamma, \tag{3.3}$$

$$\theta(0,\cdot) = \theta_0(x), \quad w(0,\cdot) = w_0(x) \quad \text{in } \Omega. \tag{3.4}$$

where Ω is a bounded domain in R^N $(N \geq 1)$, with smooth boundary $\Gamma = \partial\Omega$, $\partial I_{\theta(t,x)}$ is the subdifferential of the indicator function $I_{\theta(t,x)}(\cdot)$ on the interval $[f_a(\theta(t,x)), f_d(\theta(t,x))]$, f_a and f_d are given continuous and nondecreasing functions on R such that $f_a \leq f_d$ on R and $f(t,x)$, $g(x)$, $\theta_0(x)$, $w_0(x)$ are prescribed as data.

As well known [5, 11], (3.2) is equivalent to the hysteresis operator $F(\cdot; w_0)$:

$$w(t,x) = [F(\theta(\cdot,x); w_0(x))](t), \quad (t,x) \in Q,$$

whose input-output relation $\xi(\cdot) \to w(\cdot) = F(\xi; w_0)(\cdot)$ is illustrated in Figure 3.1 (in detail, we refer for it to [11]).

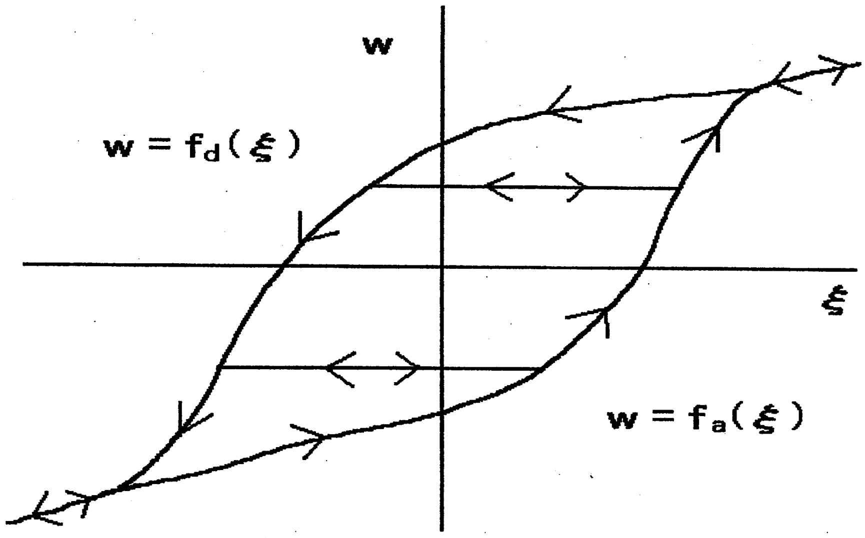

Fig.3.1

For simplicity, system (3.1)-(3.4) is denoted by (SP).

Definition 3.1. A couple of functions $\{\theta, w\}$ is called a (weak) solution of (SP) on R_+, if the following conditions (S1)-(S3) are satisfied:

(S1) $\theta \in W^{1,2}_{loc}(R_+; L^2(\Omega)) \cap L^\infty_{loc}(R_+; H^1(\Omega))$,
$w \in W^{1,2}_{loc}(R_+; L^2(\Omega))$.

(S2) $[\theta + w]_t - \Delta\theta = f(t,x)$ in $H^{-1}(\Omega)$ for a.e. $t \geq 0$ and

$$\theta(t)|_\Gamma = g \text{ on } \Gamma \quad \text{(in the sense of traces) for all } t \in R_+.$$

(S3) There exists a function $\xi \in L^2_{loc}((0,+\infty); L^2(\Omega))$ such that

$$\xi(t,x) \in \partial I_{\theta(t,x)}(w(t,x)) \quad \text{for a.e. } t \geq 0$$

and

$$w_t(t,x) + \xi(t,x) = 0 \quad \text{for a.e. } (t,x) \in R_+ \times \Omega.$$

By [8; Theorems 2.1, 5.1], an existence-uniqueness result was obtained for the Cauchy problem of (SP) as well as the existence of a periodic solution for (SP). Also the equilibrium stability and periodic stability of the solution $\{\theta, w\}$ of (SP) were discussed. In particular, in case $f(t,\cdot)$ is periodic in time, it was proved that the function θ is asymptotically periodic, but the asymptotically periodicity of the function w has not been proved yet, in the higher dimensional case.

In this section we give a proof of the asymptotically periodicity of w, too, by applying Theorem 2.1, which is an improvement of [8; Theorem 6.2]. Our result is mentioned below.

Theorem 3.1. *Let $0 < T_0 < +\infty$, $g \in H^{\frac{1}{2}}(\Gamma)$, $\theta_0 \in H^1(\Omega)$ with $\theta_0|_\Gamma = g$ a.e. on Γ, $w_0 \in L^2(\Omega)$ with $f_a(\theta_0) \leq w_0 \leq f_d(\theta_0)$ a.e. on Ω and $f = f^1 + f^2$ with $f^1 \in L^2_{loc}(R_+; L^2(\Omega))$ and $f^2 \in W^{1,1}_{loc}(R_+; H^{-1}(\Omega))$. Suppose that*

$$f(t) = f(t + T_0) \text{ in } L^2(\Omega) + H^{-1}(\Omega) \quad \text{for a.e. } t \in R_+,$$

and there are two functions f_*, $f^* \in H^{-1}(\Omega)$ such that

$$f_* \leq f(t) \leq f^* \text{ in } H^{-1}(\Omega) \quad \text{for a.e. } t \in R_+.$$

Then for any solution $\{\theta, w\}$ of (SP) associated with initial data $\{\theta_0, w_0\}$, there exists a T_0-periodic solution $\{\theta_p, w_p\}$ of (SP) such that

$$\theta(t,x) - \theta_p(t,x) \longrightarrow 0 \text{ for a.e. } x \in \Omega, \tag{3.5}$$

$$w(t,x) - w_p(t,x) \longrightarrow 0 \text{ for a.e } x \in \Omega, \tag{3.6}$$

as $t \to +\infty$.

By using Theorem 2.1 and the following lemma, we can prove Theorem 3.1.

Lemma 3.1. *Suppose all the assumption of Theorem 3.1 hold. Then, for any solution*

$\{\theta, w\}$ of (SP) with initial data $\{\theta_0, w_0\}$, there exist a finite time $t_0 \in R_+$ and $f^\infty, f_\infty \in H^{-1}(\Omega)$ such that

$$f_\infty \le f_* \le f(t_0) \le f^* \le f^\infty \ \textit{in}\ H^{-1}(\Omega),$$

and

$$z_\infty \le \theta(t_0) \le z^\infty \quad \textit{and} \quad f_a(z_\infty) \le w(t_0) \le f_d(z^\infty) \quad \textit{a.e. on}\ \Omega, \tag{3.7}$$

where z_∞ and z^∞ are the solutions of the following stationary problems:

$$-\Delta z_\infty = f_\infty \ \textit{in}\ H^{-1}(\Omega), \quad z_\infty|_\Gamma = g \quad \textit{a.e. on}\ \Gamma;$$

$$-\Delta z^\infty = f^\infty \ \textit{in}\ H^{-1}(\Omega), \quad z^\infty|_\Gamma = g \quad \textit{a.e. on}\ \Gamma.$$

4. Application to double obstacle problems for PDEs

Let us consider a double obstacle problem for a PDE of the form

$$u_t - \kappa\Delta u + \partial I_{K(\cdot)}(u) + g(u) \ni 0 \quad \text{in } Q := R_+ \times \Omega, \tag{4.1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Sigma := R_+ \times \Gamma, \tag{4.2}$$

where Ω is a bounded domain in R^N $(1 \le N < +\infty)$, with smooth boundary $\Gamma := \partial\Omega$, for each $t \in R_+ := [0, +\infty)$, $g(u) = u^3 - u$ and given obstacle functions σ_0, $\sigma_1 \in W^{1,2}_{loc}(R_+)$,

$$K(t) := \left\{ z \in L^2(\Omega);\ \sigma_0(t) \le z \le \sigma_1(t) \quad \text{a.e. on } \Omega \right\},$$

$\partial I_{K(t)}$ is the subdifferential of the indicator function $I_{K(t)}$ on $L^2(\Omega)$ defined by

$$I_{K(t)}(z) := \begin{cases} 0, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise.} \end{cases}$$

For simplicity, we denote (4.1)-(4.2) by (P) and (4.1)-(4.2) with T_0-periodic condition $u(t) = u(t + T_0)$ by (PP).

We assume further that the obstacle functions σ_i, $i = 1, 2$, satisfy

$$\sigma_0(t) \le \sigma_1(t),\ \sigma_0(t) = \sigma_0(t + T_0) \text{ and } \sigma_1(t) = \sigma_1(t + T_0), \quad \forall t \in R_+.$$

Definition 4.1. (1) A function $u : R_+ \to L^2(\Omega)$ is called a solution of (P), if it satisfies the following conditions (P1)-(P3):

(P1) $u \in C(R_+; L^2(\Omega)) \cap L^2_{loc}((0, +\infty); H^1(\Omega)) \cap W^{1,2}_{loc}((0, +\infty); L^2(\Omega))$.

(P2) $u(t) \in K(t)$ for all $t \in R_+$.

(P3) There is a function $\xi \in L^2_{loc}(R_+; L^2(\Omega))$, with $\xi(t) \in \partial I_{K(t)}(u(t))$ for a.e. $t \in R_+$, such that

$$(u'(t) + \xi(t) + g(u(t)), z) + \int_\Omega \nabla u(t) \cdot \nabla z dx = 0$$

for all $z \in H^1(\Omega)$ and a.e. $t \in R_+$.

(2) A solution u of (P) is called that of (PP) if $u(t) = u(t + T_0)$ for all $t \in R_+$.

As is easily checked, (P) is written in the form:

$$\text{(E)} \qquad u'(t) + \partial\varphi^t(u(t)) + g(u(t)) \ni 0, \quad t > 0,$$

in Hilbert space $H := L^2(\Omega)$, where $\partial\varphi^t$ is the subdifferential of time-dependent proper l.s.c. convex function $\varphi^t(\cdot)$ on H defined by

$$\varphi^t(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx & \text{if } z \in K(t) \cap H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

According to [3, 10, 13], the Cauchy problem for (P) has one and only one solution, provided that the initial value is prescribed in $K(0)$, and the T_0-periodic problem (PP) has at least one solution.

Here noted that if the initial value u_0 is constant on Ω, then the solution of (P) with $u(0, \cdot) = u_0$ is that of the scalar double obstacle problem (2.5) treated in section 2.

Now, let us consider the large time behaviour of solutions of (P). Our main theorem is stated as follows:

Theorem 4.1. *(1) Suppose that obstacle functions satisfy*

$$\sigma_0(t) \le 0 \le \sigma_1(t), \quad \forall t \in R_+.$$

Then, any solution u of (P) with initial value $u_0 \ge 0$ for a.e. on Ω or $u_0 \le 0$ for a.e. on Ω is asymptotically T_0-periodic. More precisely, one of the following three cases (i), (ii) and (iii) occurs:
(i) $u(t) - u^*(t) \to 0$ in $L^\infty(\Omega)$ as $t \to +\infty$, where u^* is the maximal T_0-periodic solution of the scalar double obstacle problem (2.5).
(ii) $u(t) - u_*(t) \to 0$ in $L^\infty(\Omega)$ as $t \to +\infty$, where u_* is the minimal T_0-periodic solution of the scalar double obstacle problem (2.5).
(iii) $u(t) \to -1$ or 0 or 1 in $L^\infty(\Omega)$ as $t \to +\infty$.
(2) Suppose that there exists $t_0 \in [0, T_0]$ such that

$$\sigma_0(t_0) > 0 \text{ or } 0 > \sigma_1(t_0).$$

Then, any solution of (P) is asymptotically T_0-periodic, namely, the following (iv) occurs:
(iv) $u(t) - u_p(t) \to 0$ in $L^\infty(\Omega)$ as $t \to +\infty$, where u_p is the unique T_0-periodic solution of (2.5).

Now, we give numerical experiences for (P) in one dimensional case,

$$u_t - \kappa u_{xx} + g(u) + \partial I_{K(t)}(u(t)) \ni 0 \quad \text{in } Q := R_+ \times (0,1), \tag{4.3}$$

$$u_x(t,0) = u_x(t,1) = 0 \quad \text{for } t > 0. \tag{4.4}$$

Here we consider the following cases.

Case 1. We assume that

$$\sigma_0(t) \leq -1 \quad \text{and} \quad 1 \leq \sigma_1(t), \quad \forall t \in R_+.$$

In this case, (iii) of Theorem 4.1 holds. If $u_0 \equiv 0$ on Ω, then the solution $u \equiv 0$ for all $(t,x) \in Q$. In the initial data $u_0 \leq 0$ a.e. on Ω with $\int_\Omega u_0(x)dx < 0$, the solution u of (4.3)-(4.4) with initial value u_0 converges to -1 in $L^\infty(\Omega)$ as $t \to +\infty$.

In the initial data $u_0 \geq 0$ a.e. on Ω with $\int_\Omega u_0(x)dx > 0$, the solution u of (4.3)-(4.4) with initial value u_0 converges to 1 in $L^\infty(\Omega)$ as $t \to +\infty$. In this case, the behaviour of solution u of (4.3)-(4.4) is illustrated in Fig.4.1

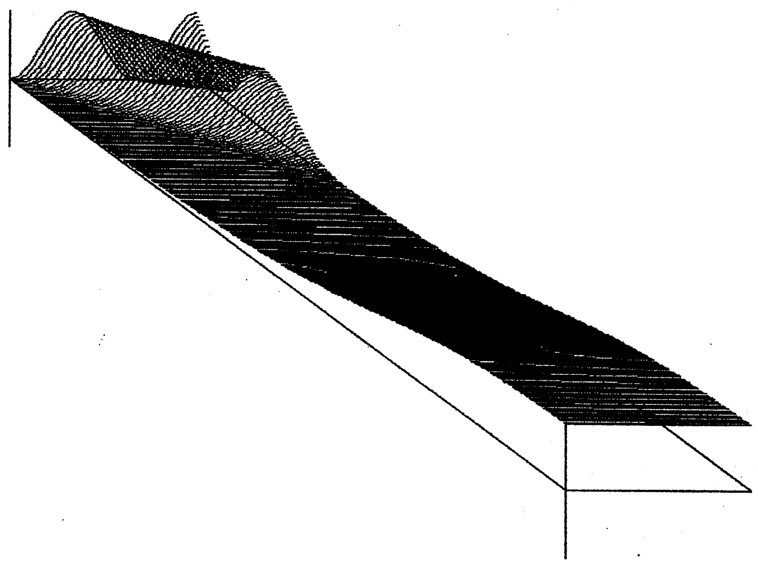

Fig.4.1

Case 2. Assume that $\sigma_0(t) < 0 \leq \sigma_1(t)$ for any $t \in R_+$,

$$-1 < \sigma_0(t_0) \quad \text{for some } t_0 \in R_+ \quad \text{and} \quad \sigma_1(t_1) < 1 \quad \text{for some } t_1 \in R_+.$$

If $u_0 \equiv 0$ on Ω, then the solution $u \equiv 0$ for all $(t,x) \in Q$.

In case $u_0 \geq 0$ a.e. on Ω with $\int_\Omega u_0(x)dx > 0$, the solution u of (4.3)-(4.4) with initial value u_0 converges to $u^*(t)$ in $L^\infty(\Omega)$ as $t \to +\infty$, where u^* is the maximal T_0-periodic solution of the scalar double obstacle problem (2.5).

In case $u_0 \leq 0$ a.e. on Ω and $\int_\Omega u_0 dx < 0$, the solution u of (4.3)-(4.4) with initial value u_0 converges to $u_*(t)$ in $L^\infty(\Omega)$ as $t \to +\infty$, where u_* is the minimal T_0-periodic solution of the scalar double obstacle problem (2.5).

In Case 2, the behaviour of solution u of (P) is illustrated in Fig.4.2-4.3.

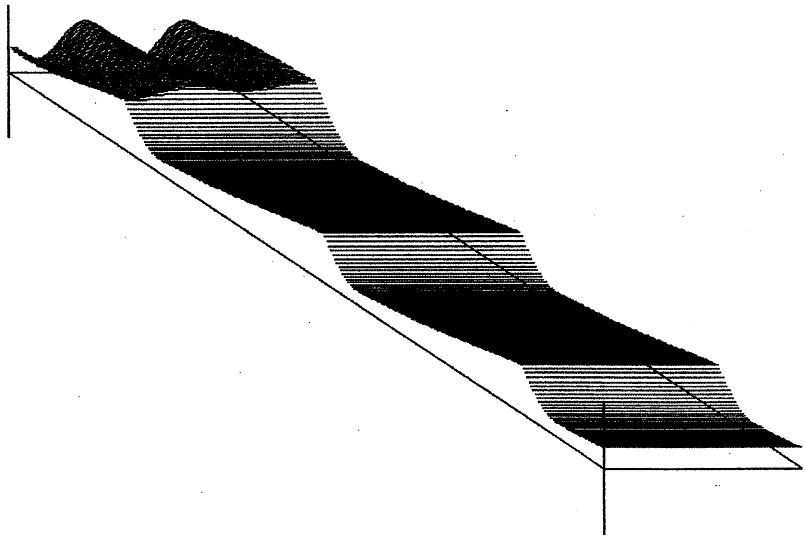

Fig.4.2

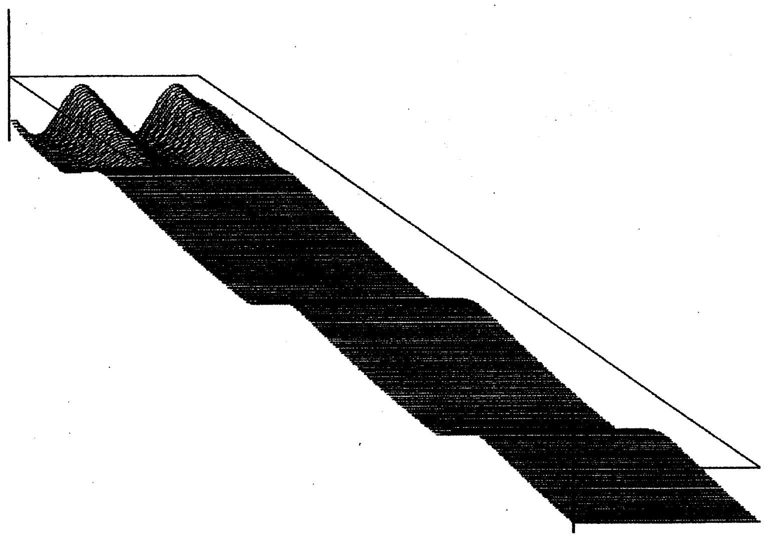

Fig.4.3

<u>Case 3.</u> Assume that

$$\sigma_0(t_0) \geq 0 \quad \text{for some } t_0 \in R_+ \quad \text{and} \quad \sigma_1(t_1) \leq 0 \quad \text{for some } t_1 \in R_+.$$

In this case, the scalar double obstacle problem (2.5) has a unique T_0-periodic solution u_p. Hence we see that any solution u of (4.3)-(4.4) converges to u_p in $L^\infty(\Omega)$ as $t \to +\infty$.

The behaviour of solution u of (4.3)-(4.4) is illustrated in the Fig.4.4.

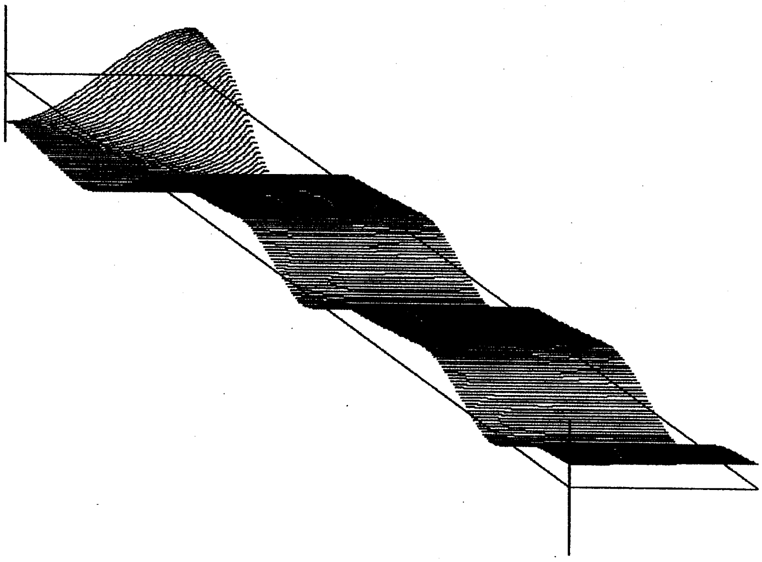

Fig.4.4

Remark. (1) In Case 1, N. Chafee and E. F. Infante [1] showed that any solution u of (4.3)-(4.4) converges to some stationary solution of (4.3)-(4.4) in one dimensional case. But in higher dimensional case, the asymptotic behaviour of any solution u is still open.

(2) In Case 2, if the initial function u_0 changes the sign, we do not know if the solution u is asymptotically T_0-periodic or not. The behaviour of solution u of (4.3)-(4.4) is illustrated in Fig.4.5. Our numerical experiences suggest the T_0-periodicity of any solution.

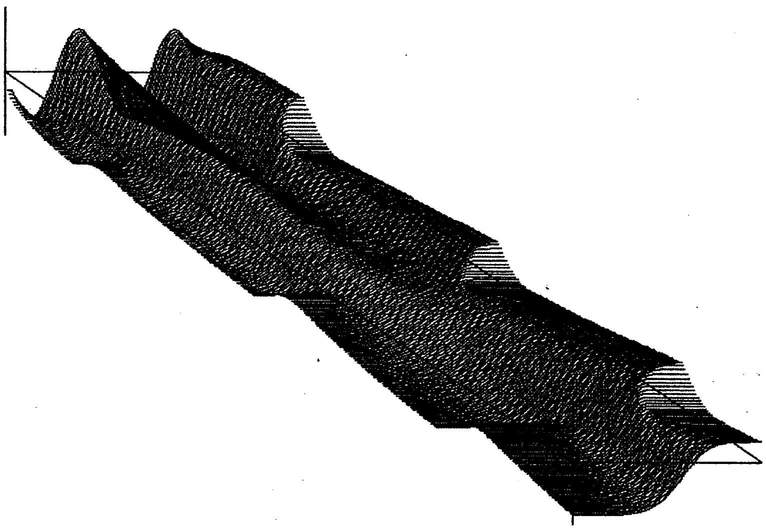

Fig.4.5

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